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A SHORT PROOF OF A THEOREM ON COMPLEX LINDENSTRAUSS SPACES

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ABSTRACT

A short proof is given of a theorem of Lima which asserts that a Banach space A with the property that every family of four balls in A with the weak intersection property has a non-empty intersection, is a Lindenstrauss space, i.e. A^* is isometric to an $L^1(\mu)$-space.

1. Introduction

Let A be a Banach space defined over the complex scalars C. If $B(x,r)$ denotes the closed ball (in A) with centre at x and of radius r, we say that A is an $E(4)$ space if, given a family $\{B(x_k,r_k)\}_{k=1}^4$ of 4 balls with the *weak intersection property*, i.e.

$$\bigcap_{k=1}^{4} B(\phi(x_k),r_k)\neq\varnothing \qquad \forall\phi\in A^*,\quad \|\phi\|\leqq 1,$$

then

$$\bigcap_{k=1}^{4} B(x_k,r_k)\neq\varnothing$$

It was conjectured by Hustad [3], and proved by Lima [4], that the dual of an $E(4)$ space is isometric to an $L^1(\mu)$-space.

The object of this note is to provide a quick proof of this result using the oft-quoted Hirsberg–Lazar criterion [2] for closed subspaces of $C_C(X)$ (containing the constants and separating the points of X, a compact Hausdorff space) to be Lindenstrauss spaces, and the following property of $E(4)$ spaces (proved in [4] for a formally larger class of spaces):

THEOREM 1.1. *Let A be a complex Banach space. Denote by H^4 (A^*) the subspace*

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$$\left\{(x_1, x_2, x_3, x_4) \in (A^*)^4 : \sum_{k=1}^{4} x_k = 0\right\}$$

of $(A^*)^4$, *the* 4-*fold product of* A^* *with itself, where* $(A^*)^4$ *is equipped with the norm*

$$\|(x_1, x_2, x_3, x_4)\| = \sum_{k=1}^{4} \|x_k\|.$$

If A *is an* $E(4)$-*space, then every element* (x_1, x_2, x_3, x_4) *of* $H^4(A^*)$ *can be expressed as* $\Sigma_{j=1}^4 (y_{j1}, y_{j2}, y_{j3}, y_{j4})$ *where*

(i) $\|x_k\| = \Sigma_{j=1}^4 \|y_{jk}\|$ *for each* k,

(ii) *each* $(y_{j1}, y_{j2}, y_{j3}, y_{j4}) \in H^4(A^*)$ *and has at most* 3 *non-zero components.*

We should remark that the proof in [4] of the result that $E(4)$ spaces are L^1-preduals, is quite different and does not use the Hirsberg–Lazar theorem quoted above. An alternative derivation has been suggested by Lima [5] from the proof of his result that $E(3)$ spaces are also L^1-preduals but the arguments in [5] are quite difficult.

2. Main theorem

Let A be an $E(4)$ space. By the device employed in [5, page 339], it is no loss of generality to assume that A is a closed subspace (in the supremum norm) of $C_C(X)$, for some compact Hausdorff space X, that $1 \in A$ and that A separates the points of X. Let

$$S = \{\phi \in A^* : \|\phi\| = \phi(1) = 1\}$$

be the state space of A. By theorem 2 in [2], it is enough to verify that $\operatorname{conv}(S \cup -iS)$ is a (Choquet) simplex, and this will follow immediately once we have checked that (i) S is split in $\operatorname{conv}(S \cup -iS)$ and (ii) S is a simplex.

Proof of (i). Let

$$\lambda_1 p_1 - (1-\lambda_1) i q_1 = \lambda_2 p_2 - (1-\lambda_2) i q_2$$

where $0 < \lambda_1, \lambda_2 < 1$, and $p_1, p_2, q_1, q_2 \in S$. As $1 \in A$, S is always a "parallel" face of conv $(S \cup -iS)$, i.e. $\lambda_1 = \lambda_2$, and it suffices to check that $p_1 = p_2$.

By Theorem 1.1, we can write

$$(\lambda_1 p_1, -\lambda_1 p_2, -(1-\lambda_1) i q_1, (1-\lambda_1) i q_2) = (0, z_{12}, z_{13}, z_{14}) + (z_{21}, 0, z_{23}, z_{24}) + (z_{31}, z_{32}, 0, z_{34}) + (z_{41}, z_{42}, z_{43}, 0),$$

where each of the terms on the right belongs to $H^4(A^*)$ and we have

$$\lambda_1 p_1 = z_{21} + z_{31} + z_{41}, \qquad \lambda_1 = \|z_{21}\| + \|z_{31}\| + \|z_{41}\|, \tag{1}$$

$$-\lambda_1 p_2 = z_{12} + z_{32} + z_{42}, \qquad \lambda_1 = \|z_{12}\| + \|z_{32}\| + \|z_{42}\|, \tag{2}$$

$$-(1-\lambda_1)iq_1 = z_{13} + z_{23} + z_{43}, \qquad 1-\lambda_1 = \|z_{13}\| + \|z_{23}\| + \|z_{43}\|, \tag{3}$$

$$(1-\lambda_1)iq_2 = z_{14} + z_{24} + z_{34}, \qquad 1-\lambda_1 = \|z_{14}\| + \|z_{24}\| + \|z_{34}\|. \tag{4}$$

Let $\tilde{S}$ be the cone generated by the face S of A_1^*, the dual ball of A. As facial cones are hereditary, we see from (1) that $z_{21}, z_{31}, z_{41} \in \tilde{S}$. Similarly, we see from (2)–(4) that $-z_{12}, -z_{32}, -z_{42} \in \tilde{S}$, $iz_{13}, iz_{23}, iz_{43} \in \tilde{S}$ and $-iz_{14}, -z_{24}, -iz_{34} \in \tilde{S}$.

On adding (3) and (4),

$$\begin{aligned}(1-\lambda_1)i(q_2-q_1) &= (z_{13}+z_{14}) + (z_{23}+z_{24}) + z_{43} + z_{34}\\ &= (-z_{12}) - z_{21} - i(iz_{43}) + i(-iz_{34})\end{aligned}$$

and evaluating these functionals at 1, we get

$$0 = \|z_{12}\| - \|z_{21}\| - i(\|z_{43}\| - \|z_{34}\|).$$

Hence

$$\begin{cases}\|z_{12}\| = \|z_{21}\|,\\ \|z_{34}\| = \|z_{43}\|.\end{cases} \tag{5}$$

Similarly, by adding (1) and (3), we have

$$\begin{cases}\lambda_1 = \|z_{31}\| + \|z_{42}\|,\\ 1-\lambda_1 = \|z_{13}\| + \|z_{24}\|,\end{cases} \tag{6}$$

and on adding (1) and (4), we get

$$\begin{cases}\lambda_1 = \|z_{32}\| + \|z_{41}\|,\\ 1-\lambda_1 = \|z_{14}\| + \|z_{23}\|.\end{cases} \tag{7}$$

From (6),

$$\begin{aligned}1 &= \|z_{13}\| + \|z_{24}\| + \|z_{31}\| + \|z_{42}\|\\ &= \|z_{13}\| + \|z_{21}+z_{23}\| + \|z_{31}\| + \|z_{41}+z_{43}\|\\ &\leqq \|z_{13}\| + (\|z_{21}\| + \|z_{23}\|) + \|z_{31}\| + (\|z_{41}\| + \|z_{43}\|)\\ &= \lambda_1 + (1-\lambda_1) \qquad \text{(from the second equations in (1) and (3))}\\ &= 1.\end{aligned}$$

It follows that

$$\|z_{21}+z_{23}\| = \|z_{21}\| + \|z_{23}\|,$$

$$\|z_{41}+z_{43}\| = \|z_{41}\| + \|z_{43}\|,$$

and hence

$$\begin{cases} \|z_{42}\| = \|z_{41}\| + \|z_{43}\|, \\ \|z_{24}\| = \|z_{21}\| + \|z_{23}\|, \end{cases} \tag{8}$$

Equating (7) and (2),

$$\lambda_1 = \|z_{41}\| + \|z_{32}\| = \|z_{12}\| + \|z_{32}\| + \|z_{42}\|$$

and we have, from (8),

$$\begin{aligned} \|z_{41}\| &= \|z_{12}\| + \|z_{42}\| \\ &= \|z_{12}\| + \|z_{41}\| + \|z_{43}\|. \end{aligned}$$

Thus, $z_{12} = z_{43} = 0$ and (5) gives that $z_{21} = z_{34} = 0$. Therefore

$$\lambda_1 p_1 = z_{31} + z_{41},$$

$$-\lambda_1 p_2 = z_{32} + z_{42},$$

giving that

$$\begin{aligned} \lambda_1(p_1 - p_2) &= (z_{31}+z_{32}) + (z_{41}+z_{42}) \\ &= -z_{34} - z_{43} \\ &= 0, \end{aligned}$$

and we can conclude that $p_1 = p_2$ as required.

Proof of (ii). S being w* compact, the cone $\tilde{S}$ is locally compact. Moreover, as a simple consequence of Theorem 1.1 (see lemma 2.1 in [4]), $\tilde{S}$ has the Riesz decomposition property and hence the directed vector space $\tilde{S}-\tilde{S}$ has the Riesz interpolation property by proposition II.3.1 in [1]. By proposition II.3.2. in [1], $\tilde{S}-\tilde{S}$ is a vector lattice, i.e. S is a simplex.

References

1. E. Alfsen, *Compact convex sets and boundary integrals*, Ergebnisse Math. Grenzgebiete, Bd. 57, Springer-Verlag, Berlin and New York, 1971.

2. B. Hirsberg and A. Lazar, *Complex Lindenstrauss spaces with extreme points*, Trans. Amer. Math. Soc. **186** (1973), 141–150.

3. O. Hustad, *Intersection properties of balls in complex Banach spaces whose duals are L_1-spaces*, Acta Math. **132** (1974), 283–313.

4. A. Lima, *Complex Banach spaces whose duals are L_1-spaces*, Israel J. Math. **24** (1976), 59–72.

5. A. Lima, *An application of a theorem of Hirsberg and Lazar*, Math. Scand. **38** (1976), 325–340.

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